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A method for controlling the motion of a robot snakeboarder☆

Yu. F. Golubev

Moscow

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Abstract

The problem of planning the motions of a robot snakeboarder consisting of a snakeboard and a flywheel mounted on it and capable of performing controlled rotation relative to the snakeboard crossbar is investigated. Programmed control of the wheel axes to provide an assigned path of motion of an arbitrary point on the crossbar is described. A control of the angular acceleration of the flywheel on the snakeboard that ensures a required course of variation of the velocity of the crossbar centre both on a horizontal plane and on an inclined plane is constructed. The problem of the maximum acceleration of the snakeboard along a "figure-of-eight" trajectory is solved.

The snakeboard was invented in 1989 by Fisher and McLeod-Smith.[1] The central part of the snakeboard is the crossbar, which rests on two pairs of wheels (trucks) at its ends (Fig. 1). The trucks can rotate relative to the crossbar about parallel axes, i.e., the truck pivot axes, which are rigidly oriented relative to the crossbar and are perpendicular to the rotation axes of the wheels. Footplates parallel to the wheel axes are mounted on the truck pivot shafts.

Owing to its design and the special coordinated action of the rider's feet and body, the snakeboard, unlike the skateboard, performs wave motion near the direction selected by the rider, enabling the rider to pick up speed without pushing a foot against the ground, even if the motion is upward along an incline. With time, an aggressive skating style accompanied by various tricks has been developed for snakeboards, and snakeboarding has become an extreme form of sport. World snakeboard championships have been held annually since 1994 (except in 1998).

When there is no slipping of the wheels relative to the supporting surface, the snakeboard is a non-holonomic mechanical system with nine degrees of freedom and four non-integrable constraints. If slipping of the wheels occurs, the system becomes more complicated because information regarding the forces that appear in the contact area between each wheel and the supporting surface is needed to close the mathematical model. When the rider standing on a snakeboard performs relative motions, the system becomes even more complicated.

It would be natural to investigate snakeboard dynamics using the methods of non-holonomic mechanics.[1,2] At the same time, it is clear that essentially all the degrees of freedom of the system experience hard servo control[3,4] on the part of the rider and that the velocity field of the crossbar is rotational around the point of intersection of the wheel axes and obeys the theorem of angular momentum about this point. From a mechanical point of view, the

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E-mail address: golubev@keldysh.ru.
[1] www.snakeboarder.com.

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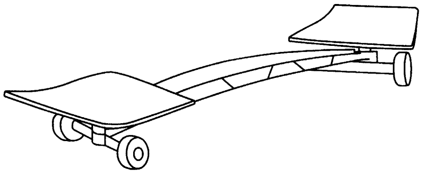

Fig. 1.

crossbar–rider system is similar to a system of bodies that perform plane-parallel motion, with the crossbar being free to rotate under the controlled action of the flywheel (the rider's body) about some centre of rotation on the supporting plane. The position of the centre of rotation, in turn, can also be altered at will. In a similar problem of control of the motion of a physical pendulum using a flywheel, the position of the centre of rotation was fixed both on the pendulum and in space in Ref. 5. An example of a more complicated system, controlled using a flywheel, was investigated in.[6]

In this paper, the servo constraints needed to control a snakeboard are written using fundamental theorems of the mechanics of the system. The approach developed enables us to deliberately alter both the magnitude and direction of the velocity and to construct the desired trajectory of any assigned point on the crossbar when all four wheels of the snakeboard are in contact with the supporting surface. The properties of the motion of such a system provide a striking illustration of the concept of an instantaneous velocity centre, as well as the theorem of angular momentum about a moving point. The non-bearing phases of the motion are not considered.

1. Statement of the problem

A mechanical system consisting of a snakeboard (Fig. 2) supplemented by a flywheel (F) which rotates about an axis parallel to the truck pivot shafts (TPS) and passes through the middle of the crossbar (C) is investigated. The presence of drives fastened to the crossbar and intended to control the rotation of the flywheel and to alter the direction of the wheel axes (WA), respectively, is assumed.

Suppose all four wheels of the snakeboard are in contact with the supporting plane. We rigidly attach the moving $Cxyz$ system of coordinates to the snakeboard crossbar. We place the origin C at the middle of the crossbar, and we direct the Cx axis along it, the Cz axis perpendicular to the supporting plane in the direction pointing away from it, where the snakeboard is located, and the Cy axis perpendicular to the crossbar and the Cz axis, so that the entire system of coordinates is right-handed. We assume that the crossbar has a length $2a$, that the left-hand wheel pair is attached to it at the point $A=(-a, 0, 0)$ and its axis makes the angle $\varphi_1+\pi/2$ with it, and that the right-hand wheel pair is attached to it at the point $D=(a, 0, 0)$ and its axis makes the angle $\varphi_2+\pi/2$ with it. Each wheel cannot slip in the direction perpendicular to the plane of the wheel. Then, the velocities of the points of articulation of the crossbar with the pivot shafts of the wheel pairs will be directed along the supporting plane perpendicularly to the wheel axes. It is required to find the laws for the autonomous control of the wheel axes and the flywheel that ensure *a priori* assigned motion of the crossbar from a state of rest.

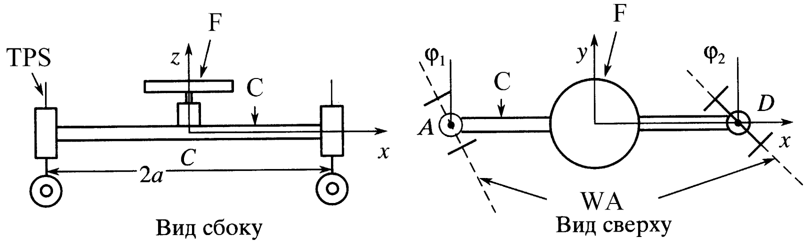

Fig. 2.

2. The kinematics of the snakeboard

The instantaneous velocity centre of the crossbar lies at the intersection of the wheel axes, and in the *Cxyz* axes it has the coordinates

$$x_\upsilon = -a\frac{\sin(\varphi_1+\varphi_2)}{\sin(\varphi_2-\varphi_1)}, \quad y_\upsilon = 2a\frac{\cos\varphi_1\cos\varphi_2}{\sin(\varphi_2-\varphi_1)} \tag{2.1}$$

As $\varphi_2 - \varphi_1 \to \pi k$ ($k = 0, \pm1, \pm2, \ldots$), the instantaneous velocity centre tends to infinity. If the coordinates of the instantaneous velocity centre are specified, the angles of rotation of the wheel axes can be found from the equalities

$$\operatorname{tg}\varphi_1 = -\frac{a+x_\upsilon}{y_\upsilon}, \quad \operatorname{tg}\varphi_2 = \frac{a-x_\upsilon}{y_\upsilon} \tag{2.2}$$

Let $O\xi\eta\zeta$ be an absolute right-handed system of coordinates for which the $O\zeta$ axis is directed along the Cz axis. In it (ξ_c, η_c, ζ_c) are the coordinates of the point C. The Cx axis makes an angle ψ with the $O\xi$ axis. Let us find the absolute coordinates of the instantaneous velocity centre

$$\xi_\upsilon = \xi_c + x_\upsilon\cos\psi - y_\upsilon\sin\psi, \quad \eta_\upsilon = \eta_c + x_\upsilon\sin\psi + y_\upsilon\cos\psi \tag{2.3}$$

The velocity field of a rigid body was constructed[7], such that the absolute velocity of the instantaneous velocity centre as a fraction of the crossbar is equal to zero. Therefore, for the velocity of the crossbar centre we have

$$\dot\xi_c = \dot\psi(x_\upsilon\sin\psi + y_\upsilon\cos\psi), \quad \dot\eta_c = \dot\psi(-x_\upsilon\cos\psi + y_\upsilon\sin\psi) \tag{2.4}$$

and for an arbitrary point (x, y, z) in the *Cxyz* axes, which is rigidly attached to the crossbar, the absolute velocity is given by the equations

$$\dot\xi = -\dot\psi[(x-x_\upsilon)\sin\psi + (y-y_\upsilon)\cos\psi], \quad \dot\eta = \dot\psi[(x-x_\upsilon)\cos\psi - (y-y_\upsilon)\sin\psi] \tag{2.5}$$

For fixed values of the angles φ_1 and φ_2, both the relative position of the instantaneous velocity centre in the *Cxyz* axes and its absolute position in the $O\xi\eta\zeta$ axes are maintained. If the angles φ_1 and φ_2 depend on time, the instantaneous velocity centre will be displaced, and it will describe a moving centroid in the *Cxyz* axes and a fixed centroid in the $O\xi\eta\zeta$ axes. The velocity of the instantaneous velocity centre over the moving centroid is expressed by the formulae

$$\dot x_\upsilon = \frac{a(\dot\varphi_2\sin2\varphi_1 - \dot\varphi_1\sin2\varphi_2)}{\sin^2(\varphi_2-\varphi_1)}, \quad \dot y_\upsilon = \frac{2a(\dot\varphi_1\cos^2\varphi_2 - \dot\varphi_2\cos^2\varphi_1)}{\sin^2(\varphi_2-\varphi_1)}$$

and the components of the velocity of the instantaneous velocity centre over the fixed centroid has the form

$$\dot\xi_\upsilon = \dot x_\upsilon\cos\psi - \dot y_\upsilon\sin\psi, \quad \dot\eta_\upsilon = \dot x_\upsilon\sin\psi + \dot y_\upsilon\cos\psi \tag{2.6}$$

If the velocities $\dot x_\upsilon$ and $\dot y_\upsilon$ are known, the angular velocities of the wheel axes are given by the equations

$$\dot\varphi_1 = \frac{(a+x_\upsilon)\dot y_\upsilon - y_\upsilon\dot x_\upsilon}{(a+x_\upsilon)^2 + y_\upsilon^2}, \quad \dot\varphi_2 = -\frac{(a-x_\upsilon)\dot y_\upsilon + y_\upsilon\dot x_\upsilon}{(a-x_\upsilon)^2 + y_\upsilon^2} \tag{2.7}$$

By assigning the centroids, we can generate the required motion of any point $B = (x, 0, 0)$ on the crossbar. When $x = 0$, point B coincides with point C, and when $x = \pm a$, point B coincides with one of the ends of the crossbar.

2.1. We will require, for example, that point B should move parallel to the $O\xi$ axis.

According to (2.5), this means that the following condition should hold

$$\dot\eta = \dot\psi[(x-x_\upsilon)\cos\psi + y_\upsilon\sin\psi] = 0$$

Assuming that $\dot\psi \neq 0$ (motion occurs), we hence find

$$(x-x_\upsilon)\cos\psi + y_\upsilon\sin\psi = 0 \tag{2.8}$$

This equality still does not uniquely define the fixed centroid.

In particular, we can additionally require that the crossbar moves translationally. We introduce the notation

$$\rho_\upsilon^2 = x_\upsilon^2 + y_\upsilon^2 \tag{2.9}$$

For the square of the distance from point C to the instantaneous velocity centre. Translational motion of the crossbar with velocity υ is obtained for $\rho_\upsilon \to \infty$, $\dot\psi \to 0$, for which $\dot\psi\rho_\upsilon = \upsilon$. Then, from equalities (2.8) and (2.2) it follows that

$$\lim\frac{x_\upsilon}{y_\upsilon} = \mathrm{tg}\psi, \quad \lim\mathrm{tg}\varphi_1 = \lim\mathrm{tg}\varphi_2 = -\mathrm{tg}\psi \quad \text{as} \quad \rho_\upsilon \to \infty$$

Stated differently, the wheel axes must be parallel to the $O\eta$ axis.

Another additional condition can be adopted by assigning, for example, the distance $\rho_\upsilon = \sqrt{x_\upsilon^2 + y_\upsilon^2}$ from point C to the instantaneous velocity centre. Then, along with (2.8) we obtain

$$x_\upsilon = \frac{x \pm \mathrm{tg}\psi\sqrt{\rho_\upsilon^2(1+\mathrm{tg}^2\psi) - x^2}}{1+\mathrm{tg}^2\psi}, \quad y_\upsilon = \frac{-x\,\mathrm{tg}\psi \pm \sqrt{\rho_\upsilon^2(1+\mathrm{tg}^2\psi) - x^2}}{1+\mathrm{tg}^2\psi} \tag{2.10}$$

These expressions have meaning for any value of ψ when $\rho_\upsilon^2 \geq x^2$. The equation of the moving centroid is given by (2.9), and for a constant value of ρ_v it corresponds to a circle. The fixed centroid is defined by the equalities

$$\xi_\upsilon = \xi_c + x\cos\psi, \quad \eta_\upsilon = \eta_c \pm \sqrt{\rho_\upsilon^2 - x^2\cos^2\psi}$$

In this case, point B will move rectilinearly along the $O\xi$ axis with velocity

$$\dot\xi_b = -\dot\psi(x\sin\psi + \eta_c - \eta_\upsilon)$$

The equations presented for the rectilinear motion of point B are simplified considerably when $x = 0$, i.e., when point B coincides with point C. Then, when the signs are chosen appropriately, we obtain

$$x_\upsilon = \pm\rho_\upsilon\sin\psi, \quad y_\upsilon = \pm\rho_\upsilon\cos\psi, \quad \xi_\upsilon = \xi_c, \quad \eta_\upsilon = \eta_c \pm \rho_\upsilon, \quad \dot\xi_c = \pm\dot\psi\rho_\upsilon \tag{2.11}$$

We will chose the plus sign on the right-hand sides of formulae (2.11). Then, for constant ρ_v the fixed centroid is a straight line parallel to the $O\xi$ axis. Point C is at the distance ρ_v from it in the direction of the origin of coordinates, the absolute vertical coordinate of point C remains constant, and for a positive value of $\dot\psi$, the absolute horizontal coordinate of point C increases.

Similarly, if we take the minus sign on the right-hand sides of formulae (2.11), for constant ρ_v the fixed centroid is a straight line parallel to the $O\xi$ axis, as in the previous case. However, in this case this centroid is at the distance ρ_v from point C in the direction of the origin of coordinates O. The absolute vertical coordinate of point C remains constant, and for a positive value of $\dot\psi$ the absolute horizontal coordinate of point C decreases.

For constant ρ_v the velocity of point C is proportional to $\dot\psi$. In order for the direction of motion of point C to remain unchanged when the sign of $\dot\psi$ changes, the sign on the right-hand sides of formulae (2.11) should alternate at the time when $\dot\psi = 0$. The motion of point C will be strictly in one direction if the sign reversal operation is performed almost instantaneously. In any case, however, the motion of point C will occur with stops when $\dot\psi = 0$.

It is possible to avoid the stopping of point C when $\dot\psi = 0$. To achieve this, the instantaneous velocity centre must tend to infinity at times corresponding to $\dot\psi = 0$. For example, taking into account that $x = 0$, we can stipulate that

$$\rho_\upsilon = 2a/(\kappa\dot\psi) \tag{2.12}$$

where κ is a constant coefficient that has the dimension of time. Thus, for any law $\psi = \psi(t)$ that defines the angular motion of the crossbar, where t is the time and the existence of the derivative $\dot\psi$ is assumed, point C will undergo linear motion parallel to the $O\xi$ axis with a constant velocity, whose direction depends on the sign of the coefficient κ; the smaller the value of $|\kappa|$, the greater the velocity of point C.

2.2. Let us consider motion for which point A is displaced strictly along the $O\xi$ axis and point D is displaced strictly along the $O\eta$ axis. Then point C will be forced to move along an arc of a circle of radius a, whose centre is at point O.

In this case,

$$\varphi_1 = -\psi, \quad \varphi_2 = \pi/2 - \psi, \quad x_\upsilon = -a\cos 2\psi, \quad y_\upsilon = a\sin 2\psi$$

As would be expected, the moving centroid is a circle of constant radius $\rho_v = a$, whose centre is at point C. Furthermore,

$$\xi_c = -a\cos\psi, \quad \eta_c = a\sin\psi, \quad \xi_\upsilon = -2a\cos\psi, \quad \eta_\upsilon = 2a\sin\psi$$

and the fixed centroid is a circle of constant radius $2a$, whose centre is at point O. The velocity of point C is expressed by the equations

$$\dot\xi_c = \dot\psi a\sin\psi, \quad \dot\eta_c = \dot\psi a\cos\psi$$

When $\dot\psi > 0$, point C moves clockwise if it is viewed from the positive direction of the $O\zeta$ axis.

2.3. Let us investigate a version of the motion in which the crossbar is directed along a tangent to the trajectory of an arbitrary point on it. We again take some point $B = (x, 0, 0)$ on the Cx axis. For simplicity we will assume that

$$(a + x)\operatorname{tg}\varphi_2 = (x - a)\operatorname{tg}\varphi_1 = (a^2 - x^2)/d \tag{2.13}$$

where d has the dimension of length. Then, from (2.1) and (2.5), we obtain

$$x_\upsilon = x, \quad y_\upsilon = d, \quad \rho_\upsilon = \sqrt{x^2 + d^2}, \quad \dot\xi_b = \dot\psi d\cos\psi, \quad \dot\eta_b = \dot\psi d\sin\psi \tag{2.14}$$

In other words, the velocity $v_b = \dot\xi_b^2 + \dot\eta_b^2$ of point B is proportional to d, and the angle of inclination of the tangent to the trajectory of point B coincides with the inclination of the crossbar to the O axis:

$$\upsilon_b = |\dot\psi d|, \quad d\eta_b/d\xi_b = \operatorname{tg}\psi$$

If d is chosen to have a constant value, the instantaneous velocity centre is a fixed point in the Cxy axes and, therefore (see (2.6)), in the $O\xi\eta$ axes. We have

$$\xi_b(t) - \xi_b(t_0) = d(\sin\psi - \sin\psi_0), \quad \eta_b(t) - \eta_b(t_0) = -d(\cos\psi - \cos\psi_0)$$

where ψ_0 is the initial value of the angle ψ. Point B moves along a circle of radius d, whose centre is at the point with coordinates

$$\xi = \xi_b(t_0) - d\sin\psi_0, \quad \eta = \eta_b(t_0) + d\cos\psi_0$$

which coincides with the instantaneous velocity centre chosen in this case, and the crossbar is oriented along the tangent to the circle at this point.

In the case where d varies, the moving centroid is a straight line passing through point B perpendicular to the crossbar. For the motion of point B with a constant velocity along an arbitrary curve, we assume that

$$d = \frac{a}{\kappa\dot\psi}, \quad \operatorname{tg}\varphi_1 = -\frac{a + x}{a}\kappa\dot\psi, \quad \operatorname{tg}\varphi_2 = \frac{a - x}{a}\kappa\dot\psi \tag{2.15}$$

Then the fixed centroid coincides with the locus of the centres of curvature of an assigned trajectory of motion of point B.

2.4. When formulae (2.15) are used, it is impossible to ensure rectilinear motion of point B. However, if the inequality $-\pi/2 < \psi < \pi/2$ is satisfied during the motion, the motion will be oscillatory and, on average, will occur along the $O\xi$ axis (hence the name “snakeboard”). The required trajectory of motion of point B along the $O\xi$ axis can be assigned by the continuously doubly differentiated function $\eta_b = \eta_b(\xi_b)$. Then we have $\cos\psi > 0$ and

$$\operatorname{tg}\psi = \eta_b', \quad \frac{\dot\psi}{\cos^2\psi} = \eta_b''\dot\xi_b, \quad \dot\psi = \frac{a\eta_b''}{\kappa(1 + \eta_b'^2)^{3/2}}$$

where the prime denotes differentiation with respect to ξ_b. For example, we can take the relation

$$\eta_b = \alpha\sin(\omega[\xi - \xi_b(t_0)]) + \eta_b(t_0)$$

For small values of the coefficients α and ω, such a trajectory will differ only slightly from a straight line parallel to the $O\xi$ axis.

A trajectory of point B of fairly arbitrary form in a plane can be specified parametrically by

$$\xi_b = \xi_b(p), \quad \eta_b = \eta_b(p) \tag{2.16}$$

where p is a parameter of the trajectory. To determine the value of $\dot\psi$ corresponding to such motion, we use Eqs. (2.14) and (2.15). Then

$$\xi'_b\dot p = \frac{a}{\kappa}\cos\psi, \quad \eta'_b\dot p = \frac{a}{\kappa}\sin\psi, \quad \mathrm{tg}\psi = \frac{\eta'_b}{\xi'_b}, \quad \dot p = \frac{a}{\kappa\sqrt{\xi_b'^2+\eta_b'^2}}, \quad \dot\psi = \frac{a(\eta''_b\xi'_b - \eta'_b\xi''_b)}{\kappa(\xi_b'^2+\eta_b'^2)^{3/2}}$$

where the prime denotes differentiation with respect to p. For the required motion to be possible, the initial conditions must correspond to the conditions stipulated by (2.15). In particular, we should have

$$\mathrm{tg}\psi(t_0) = \eta'_b(p(t_0))/\xi'_b(p(t_0))$$

Let the crossbar be oriented relative to the $O\xi$ axis and make an angle ψ_0 with it. Also let the standard equations of the trajectory of point B be specified in some rectangular system of coordinates $O'\tilde\xi\tilde\eta$ by the equations

$$\tilde\xi = \tilde\xi(p), \quad \tilde\eta = \tilde\eta(p)$$

The transformation of coordinates

$$\xi = \xi_0 + \tilde\xi\cos\alpha - \tilde\eta\sin\alpha, \quad \eta = \eta_0 + \tilde\xi\sin\alpha + \tilde\eta\cos\alpha \tag{2.17}$$

for which the equalities

$$\xi_b(t_0) = \xi_0 + \tilde\xi(0)\cos\alpha - \tilde\eta(0)\sin\alpha, \quad \eta_b(t_0) = \eta_0 + \tilde\xi(0)\sin\alpha + \tilde\eta(0)\cos\alpha$$

$$\cos\psi_0 = \frac{\tilde\xi'(0)\cos\alpha - \tilde\eta'(0)\sin\alpha}{\sqrt{\tilde\xi'^2(0)+\tilde\eta'^2(0)}}, \quad \sin\psi_0 = \frac{\tilde\xi'(0)\sin\alpha + \tilde\eta'(0)\cos\alpha}{\sqrt{\tilde\xi'^2(0)+\tilde\eta'^2(0)}}$$

hold for $p = 0$, uniquely defines the functions (2.16) that match the initial conditions with the assigned crossbar position. For example, we can take

an astroid: $\tilde\xi_b = r\cos^3 p, \quad \tilde\eta_b = r\sin^3 p$
a cycloid: $\tilde\xi_b = r(p - \sin p), \quad \tilde\eta_b = r(1 - \cos p)$
a figure of eight: $\tilde\xi_b = r\cos p, \quad \tilde\eta_b = r\sin 2p$
an n-leaved rose: $\tilde\xi_b = r\sin(6-n)p\cos p, \quad \tilde\eta_b = r\sin(6-n)p\sin p, \quad n = 3, 4$

and other beautiful curves.

3. The angular momentum equation

We recall that the snakeboard is equipped with a flywheel, whose axis passes through point C parallel to the direction of the pivot shafts of the wheelpairs (Fig. 2). The flywheel is rotated by a motor mounted on the crossbar. The angle of rotation of the flywheel relative to the crossbar is denoted by φ. Similarly, the wheel pairs are rotated by corresponding motors, which are also located on the crossbar. We shall assume that there is no rolling resistance of the wheels over the supporting plane and that the system has the property of mass symmetry about point C. This means that the centres of mass of the crossbar and the flywheel are located at point C and that the masses and moments of inertia of the wheel pairs are identical. Let the $O\zeta$ axis make an angle ϑ with the vertical so that the gravity force of the system, applied at point C, in the $O\xi\eta\zeta$ axes has the form

$$\mathbf{P} = -Mg(0, \sin\vartheta, \cos\vartheta)$$

where M is the mass of the entire system and g is the acceleration due to gravity. When there is no longitudinal rolling resistance, reactions parallel to the supporting plane are directed exactly into the instantaneous velocity centre of the crossbar, impeding displacement of the wheels along their axes. Therefore, taking into account relations (2.3), we can write the projection of the equation describing the variation of the angular momentum of the snakeboard together with the flywheel relative to the instantaneous velocity centre of the crossbar onto the $O\zeta$ axis in the form[7]

$$\frac{dK}{dt} - M\dot{\psi}(x_v\dot{x}_v + y_v\dot{y}_v) = (x_v\cos\psi - y_v\sin\psi)Mg\sin\vartheta \tag{3.1}$$

Here K is the angular momentum of the entire system about the axis passing through the instantaneous velocity centre parallel to the $O\zeta$ axis, and the expression in parentheses on the left-hand side of Eq. (3.1) is the projection of the vector product of the velocity of the instantaneous velocity centre and the velocity of the centre of mass C onto the $O\zeta$ axis. By Koenig's theorem,[7] we obtain

$$K = \dot{\psi}(M\rho_v^2 + b) + J_g\dot{\varphi} + J_w(\dot{\varphi}_1 + \dot{\varphi}_2) \tag{3.2}$$

where

$$M = m + m_g + 2m_w, \quad b = 2m_w a^2 + J + J_g + 2J_w$$

m is the mass of the crossbar, m_g is the mass of the flywheel, m_w is the mass of a wheel pair, J is the moment of inertia of the crossbar together with parts of the wheel systems rigidly attached to it about to the $C\zeta$ axis, J_g is the moment of inertia of the flywheel about the same axis, and J_w is the moment of inertia of a wheel pair about its axis. It is easy to see that

$$x_v\dot{x}_v + y_v\dot{y}_v = \frac{d}{dt}\left(\frac{\rho_v^2}{2}\right)$$

It follows from the formulae (2.1) that ρ_v^2 is a function of the angles φ_1 and φ_2, which are realized by the corresponding drives. When the servo constraints[3] are selected in the form of functions that describe the dependence of ρ_v on the angles and angular velocities of the crossbar and the flywheel, Eq. (3.1) defines the mutual influence of the angular accelerations of the flywheel and the crossbar. By assigning the required motion of the crossbar in this case, we can obtain the variation of the angular acceleration of the flywheel that realizes this motion.

4. Programmed control

We will consider several examples in this section, ignoring the influence of the transients that occur when the servo constraints[3] are imposed.

4.1. Servo constraints of the form $p_v = p_v(\dot{\psi})$

We set[2] $\vartheta = 0$ (the supporting plane is horizontal). Equation (3.1) then becomes

$$\frac{d}{dt}\left(K - \frac{M\dot{\psi}}{2}\rho_v^2\right) + \frac{M\ddot{\psi}}{2}\rho_v^2 = 0$$

Hence it follows that if ρ_v depends only on $\dot{\psi}$, the angular momentum equation has a first integral

$$K - \frac{M\dot{\psi}}{2}\rho_v^2 + F(\dot{\psi}) = \sigma, \quad F(\dot{\psi}) = \frac{M}{2}\int\rho_v^2(\dot{\psi})d\dot{\psi} \tag{4.1}$$

where σ is the constant of integration.

[2] Golubev Yu.F. Planning the motions of a robot snakeboarder. Preprint of the M.V. Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, No. 65, Moscow: Institute of Applied Mathematics; 2004.

4.1a. Let the law $\psi = \psi_s(t)$ of the required angular motion of the crossbar be assigned, and let equalities (2.8) and (2.12), which stipulate that point C moves in a straight line along the $O\xi$ axis with a constant velocity, be satisfied. Then

$$\rho_\upsilon^2 = \frac{4a^2}{\kappa^2\dot\psi_s^2}, \quad F(\dot\psi_s) = -\frac{2Ma^2}{\kappa^2\dot\psi_s} \tag{4.2}$$

The integral of the angular momentum for $\psi \equiv \psi_s$ takes the form

$$b\dot\psi_s + J_g\dot\varphi_s + J_w(\dot\varphi_1 + \dot\varphi_2) = \sigma \tag{4.3}$$

This integral was obtained under the assumption that, as follows from (2.2) and (2.12), the angles of the wheel pairs satisfy the equalities

$$\operatorname{tg}\varphi_{js} = (-1)^j\frac{\kappa\dot\psi_s}{2a\cos\psi_s} - \operatorname{tg}\psi_s, \quad j = 1, 2 \tag{4.4}$$

Relations (4.3) and (4.4) taken together should be regarded as the set of servo constraints that ensure the crossbar motion assigned using the programming function $\psi = \psi_s(t)$, provided the servo constraints are satisfied at the initial instant of time. Since in reality the initial conditions can be arbitrary, the control of the motion of the flywheel and the wheel pairs must ensure the relations[3]

$$\psi \to \psi_s, \quad \varphi \to \varphi_s, \quad \varphi_j \to \varphi_{js} \quad \text{as} \quad t \to \infty \tag{4.5}$$

Here and everywhere henceforth $j = 1, 2$.

When we take into account that function $\psi_s(t)$ is assigned and known, relations (4.5) define the geometric servo constraints. To obtain the corresponding differential servo constraints, we must find

$$\dot\varphi_{js} = (-1)^j\frac{q_j\cos^2\varphi_{js}}{\cos^2\psi_s}, \quad q_j = \frac{\kappa}{2a}(\ddot\psi_s\cos\psi_s + \dot\psi_s^2\sin\psi_s) + (-1)^{j+1}\dot\psi_s \tag{4.6}$$

To construct the precise transient we also require the second derivatives of the functions (4.6):

$$\begin{aligned}\ddot\varphi_{js} &= -2\dot\varphi_{js}^2\operatorname{tg}\varphi_{js} + (-1)^j\frac{\cos^2\varphi_{js}}{\cos^3\psi_s}(\dot q_j\cos\psi_s + 2q_j\dot\psi_s\sin\psi_s)\\ \dot q_j &= \frac{\kappa}{2a}[(\dddot\psi_s + \dot\psi_s^3)\cos\psi_s + \ddot\psi_s\dot\psi_s\sin\psi_s] + (-1)^{j+1}\ddot\psi_s\end{aligned} \tag{4.7}$$

In addition, we note that the second derivative of relation (4.3) can be represented, taking relations (4.2) into account, in the equivalent form

$$\frac{d}{dt}[\dot\psi_s(M\rho_\upsilon^2 + b) + J_g\dot\varphi_s + J_w(\dot\varphi_{1s} + \dot\varphi_{2s})] - \frac{M}{2}\dot\psi_s\frac{d\rho_\upsilon^2}{dt} = 0 \tag{4.8}$$

4.1b. We will assume that the law $\psi = \psi_s(t)$ of angular motion of the crossbar is specified and that equalities (2.15) hold for it. In this case,

$$\rho_\upsilon^2 = x^2 + \frac{a^2}{\kappa^2\dot\psi_s^2}, \quad F(\dot\psi_s) = \frac{M}{2}\left(x^2\dot\psi_s - \frac{a^2}{\kappa^2\dot\psi_s}\right)$$

The integral of the angular momentum (4.1) takes the form

$$(Mx^2 + b)\dot\psi_s + J_g\dot\varphi_s + J_w(\dot\varphi_{1s} + \dot\varphi_{2s}) = \sigma \tag{4.9}$$

and the angles φ_{1s} and φ_{2s} are given by the formulae

$$\operatorname{tg}\varphi_{js} = \kappa_j\dot\psi_s, \quad \kappa_j = \kappa[(-1)^j - x/a] \tag{4.10}$$

Differentiation of the servo constraints (4.10) gives the equalities

$$\dot{\varphi}_{js} = \kappa_j \ddot{\psi}_s \cos^2\varphi_{js}, \quad \ddot{\varphi}_{js} = \kappa_j(\dddot{\psi}_s \cos^2\varphi_{js} - \ddot{\psi}_s \dot{\varphi}_{js} \sin 2\varphi_{js}) \tag{4.11}$$

The derivative of the servo constraint (4.9) can be converted to the form (4.8).

We see that the existence of not only the second derivative of the programming function $\psi_s(t)$ with respect to time but also of the third derivative is required to construct the transient by controlling the wheel pairs both in case 4.1a and in case 4.1b.

4.2. A servo constraint of the form $\rho_\upsilon = \rho_\upsilon(\varphi, \dot{\varphi})$

Let $\vartheta = 0$ as before. The angular momentum equation (3.1) can be rewritten in the form

$$\ddot{\psi}(M\rho_\upsilon^2 + b) + \frac{M}{2}\dot{\psi}\frac{d}{dt}\rho_\upsilon^2 = -J_g\ddot{\varphi} - J_w(\ddot{\varphi}_1 + \ddot{\varphi}_2)$$

We will use the fact that the left-hand side of this equation has the integrating factor $\left(\pm\sqrt{M\rho_\upsilon^2 + b}\right)^{-1}$. Taking into account the possibility of negative values of ρ_υ, we rewrite the equation in the form

$$\frac{d}{dt}(\dot{\psi}\rho_\upsilon\mu) = -\frac{J_g\ddot{\varphi} + J_w(\ddot{\varphi}_1 + \ddot{\varphi}_2)}{\rho_\upsilon\mu}, \quad \mu = \sqrt{M + b\rho_\upsilon^{-2}} \tag{4.12}$$

We constrain the motions of the wheel pairs using relations (2.13), where, for simplicity, we set $x = 0$. Then point B coincides with the middle C of the crossbar, and $\rho_\upsilon = d$, so that

$$\operatorname{tg}\varphi_{js} = (-1)^j a/\rho_\upsilon, \quad \varphi_{1s} + \varphi_{2s} \equiv 0 \tag{4.13}$$

and the angular momentum equation for the programmed motion takes the form

$$\frac{d}{dt}(\dot{\psi}\rho_\upsilon\mu) = -\frac{J_g\ddot{\varphi}}{\rho_\upsilon\mu} \tag{4.14}$$

4.2a. Under the condition $\rho_\upsilon = \rho_\upsilon(\dot{\varphi})$, Eq. (4.14) has a first integral

$$\dot{\psi}\rho_\upsilon\mu + F_1(\dot{\varphi}) = \sigma_1; \quad F_1 = J_g\int\frac{d\dot{\varphi}}{\rho_\upsilon\mu} \tag{4.15}$$

Integral (4.15) indicates that if the function φ is periodic, the velocity $\upsilon_c = \dot{\psi}\rho_\upsilon$ of the middle C of the crossbar will also be periodic. In other words, an endless increase in the velocity of point C will not occur in this case. For example, let $\rho_\upsilon = \kappa\dot{\varphi}^{-1}$. Then

$$F_1 = 2J_g(\kappa b)^{-1}\sqrt{M + b\kappa^{-2}\dot{\varphi}^2}$$

and integral (4.15) takes the form

$$\upsilon_c = -2J_g(\kappa b)^{-1} + \sigma_1(M + b\kappa^{-2}\dot{\varphi}^2)^{-1/2}, \quad \sigma_1 = 2J_g(\kappa b)^{-1}\sqrt{M}$$

4.2b. We specify a periodic law governing the variation of the angle φ using the equation

$$\ddot{\varphi} + \omega^2\varphi = 0 \tag{4.16}$$

In addition, suppose $\rho_\upsilon = \kappa_1\varphi^{-1}$. Then Eq. (4.14) takes the form

$$\frac{d}{dt}(\dot{\psi}\rho_\upsilon\sqrt{M + b\rho_\upsilon^{-2}}) = J_g\kappa_1^{-1}\omega^2\varphi^2(M + b\kappa_1^{-2}\varphi^2)^{-1/2} \tag{4.17}$$

The right-hand side of this equation is non-negative. Therefore, for $\varphi \not\equiv 0$ the velocity $v_c = \dot{\psi}\rho_v$ will increase, and, for example, for $\varphi = \alpha \cos \omega(t - t_0)$ it can be made infinitely large. To illustrate this, instead of the exact equality (4.17), let us consider the approximate equation

$$\frac{d}{dt}(\dot{\psi}\rho_v\sqrt{M}) = J_g\kappa_1^{-1}\omega^2\varphi^2M^{-1/2} \tag{4.18}$$

which characterizes the acceleration of the snakeboard at small values of the amplitude α. Let $\tau = t - t_0$, Integrating, we find

$$v_c = \frac{J_g\omega^2\alpha^2}{2\kappa_1\sqrt{M}}\left(\tau + \frac{\sin 2\omega\tau}{2\omega}\right), \quad \psi = \frac{J_g\omega\alpha^2}{2\kappa_1^2M}\left(\tau\sin\omega\tau - \frac{3\cos\omega\tau + \cos^3\omega t - 4}{3\omega}\right)$$

We see that a linear increase in the amplitude of the oscillations of the angle ψ with time, similar to the increase previously found in Ref. 2, occurs in the wheel axis control regime selected.

4.3. Consider the possibility of acceleration of the crossbar in the case where the trajectory of the middle C is specified. We shall assume that $\vartheta \neq 0$. We shall also assume that the axial line of the crossbar is directed along a tangent to the trajectory and that the wheel axes are controlled in accordance with relations (2.13), in which we set $x = 0$.

Suppose the trajectory is specified parametrically in the form

$$\xi = \xi(p), \quad \eta = \eta(p), \quad \frac{1}{\rho_v} = \frac{\xi'\eta'' - \eta'\xi''}{(\xi'^2 + \eta'^2)^{3/2}} \tag{4.19}$$

The prime denotes differentiation with respect to p. When the integrating factor is taken into account (see Section 4.2), Eq. (3.1) takes the form

$$\frac{d(\mu v_c)}{dt} = -\frac{J_g\ddot{\varphi}(\xi'\eta'' - \eta'\xi'') + Mg\eta'(\xi'^2 + \eta'^2)\sin\vartheta}{\mu(\xi'^2 + \eta'^2)^{3/2}} \tag{4.20}$$

The quantity $\mu > 0$ does not vanish at any point along the trajectory, and it will also be restricted for trajectories that do not have singularities. Hence we see that an increase in velocity along the trajectory is achieved when

$$J_g\ddot{\varphi}(\xi'\eta'' - \eta'\xi'') < -Mg\eta'(\xi'^2 + \eta'^2)\sin\vartheta \tag{4.21}$$

and that a decrease in velocity occurs when the sign of the inequality is reversed. In other words, if $\eta' < 0$, gravity starts accelerating the snakeboard when $\sin\vartheta > 0$, and to impart additional acceleration to it the angular acceleration of the flywheel must be non-positive at points where the trajectory turns to the left relative to the direction of the tangent, and it must be non-negative at points where the trajectory turns to the right. The absolute value of the angular acceleration is clearly determined by the required magnitude of the velocity increment.

Conversely, if $\eta' > 0$, gravity decelerates the snakeboard. Therefore, in accordance with inequality (4.21), a velocity increment is achieved for strictly negative (strictly positive) values of the angular acceleration of the flywheel if the trajectory is turning to the left (to the right).

When $\ddot{\varphi} \equiv 0$, the velocity will not be constant in the general case both because of the effect of gravity and because of the variation of the radius of curvature of the trajectory: $v = \sigma/\mu$, where σ is the constant of the integral of the angular momentum for $\ddot{\varphi} \equiv 0$ and $\sin\vartheta = 0$.

According to the equalities (4.13), the angles of the wheel axes are specified by the formulae

$$\operatorname{tg}\varphi_{2s} = -\operatorname{tg}\varphi_{1s} = \frac{a(\xi'\eta'' - \eta'\xi'')}{(\xi'^2 + \eta'^2)^{3/2}} \tag{4.22}$$

The variation of the parameter p is given by the equation

$$\dot{p} = \frac{v}{\sqrt{\xi'^2 + \eta'^2}} \tag{4.23}$$

Thus, the system of equations (4.20), (4.19) and (4.23) is closed.

To initiate motion, it is sufficient to attach the trajectory to the position of point C on the plane, set the angles of the wheel axes according to the required radius of curvature, and impart angular acceleration to the flywheel in accordance with inequality (4.21). The unavoidable errors in controlling the wheel axes result in distortion of the assigned trajectory of motion. However, if extremely high requirements are not imposed on the quality of the programmed motions, the trajectory of point C will have a definite similarity to the specified trajectory even in the absence of exact navigational adherence to location. We also note that a constant increase in velocity is not required for realizing complex trajectories, since slipping of the wheels relative to the supporting surface may occur.

4.3a. We specify the trajectory of point C in the form

$$\eta = \eta(p), \quad \xi = p, \quad \frac{1}{\rho_\upsilon} = \frac{\eta''}{(1+\eta'^2)^{3/2}}$$

Equation (4.2) can now be rewritten in the form

$$\frac{d(\mu\upsilon_c)}{dt} = -\frac{J_g\ddot{\varphi}\eta'' + Mg\eta'(1+\eta'^2)\sin\vartheta}{\mu(1+\eta'^2)^{3/2}} \tag{4.24}$$

The value of μ does not vanish anywhere and is restricted in magnitude when the values of η' and η'' are restricted. Therefore, to ensure an increase in the velocity of point C, it is sufficient to control the motion of the flywheel so that the following inequality holds over almost the entire assigned acceleration trajectory

$$J_g\ddot{\varphi}\eta'' < -Mg\eta'(1+\eta'^2)\sin\vartheta \tag{4.25}$$

Conversely, to reduce the velocity, it is sufficient, by controlling the flywheel, to achieve satisfaction of the opposite inequality. The angles of the wheel axes are determined by the position of point C on the trajectory

$$\mathrm{tg}\,\varphi_{2s} = -\mathrm{tg}\,\varphi_{1s} = \frac{a\eta''}{(1+\eta'^2)^{3/2}}$$

For example, let the trajectory of point C be specified in the form

$$\eta = \eta_0\sin[\nu(\xi-\xi_0)], \quad \eta' = \eta_0\nu\cos[\nu(\xi-\xi_0)], \quad \eta'' = -\nu^2\eta \tag{4.26}$$

In this case, when $\sin\vartheta > 0$, an increase in the velocity of point C will be ensured if, for example, the angular acceleration $\ddot{\varphi}$ is made non-positive (non-negative) for $\eta < 0$ and $\eta' < 0$ (for $\eta > 0$ and $\eta' > 0$) in accordance with inequality (4.25). On segments where $\eta' > 0$, the values of $\ddot{\varphi}$ cannot vanish according to (4.25) because gravity prevents acceleration on these segments. Note that the singularities associated with the sign of η' disappear for $\vartheta = 0$.

Next, let $\eta_0\nu \ll 1$. This means that the wavelength $\lambda = 2\pi/\nu$ of the trajectory significantly exceeds the amplitude of its oscillations (motion occurs almost along a straight line). Then

$$\varphi_{2s} = -\varphi_{1s} \approx -a\nu^2\eta$$

In other words, the angles the wheel axes make with the Cy axis should be small and proportional to the deviation of the trajectory from the $O\xi$ axis. For $\eta > 0$ they will, as would be expected, satisfy the inequality $\varphi_{2s} = -\varphi_{1s} < 0$, which is reversed for $\eta < 0$.

The method described for accelerating the snakeboard along the trajectory (4.26) does not provide the possibility of starting the motion from the point where $\eta = 0$, because the wheel axes are perpendicular to the crossbar and $\ddot{\varphi} = 0$ at that point. This difficulty can be avoided by slightly displacing the origin of coordinates and thereby the entire programmed trajectory in a direction parallel to the trajectory so that we would have $\eta \neq 0$ at the initial instant of time for the motion. Another technique is to set $\eta \approx (\xi - \xi_0)\eta'$ in a small neighbourhood of the special point. Then inequality (4.25) takes the form

$$-J_g\nu^2\ddot{\varphi}(\xi-\xi_0)\eta' < -Mg\eta'(1+\eta'^2)\sin\vartheta$$

from which the required sign of the initial angular acceleration of the flywheel can be found.

4.3b. Consider a trajectory of the form

$$\xi = \xi(p), \quad \eta = p, \quad \eta' = 1, \quad \frac{1}{\rho_\upsilon} = -\frac{\xi''}{(1+\xi'^2)^{2/3}} \tag{4.27}$$

Trajectories of this type can be used when $\sin\vartheta > 0$ for upward motion along an incline, which corresponds to an increase in the parameter p. Equation (4.20) takes the form

$$\frac{d(\mu\upsilon_c)}{dt} = \frac{J_g\ddot{\varphi}\xi'' - Mg(1+\xi'^2)\sin\vartheta}{\mu(\xi'^2+\eta'^2)^{3/2}} \tag{4.28}$$

Thus, when $\sin\vartheta > 0$, for upward acceleration of the snakeboard, the inequality

$$J_g\ddot{\varphi}\xi'' > Mg(1+\xi'^2)\sin\vartheta \tag{4.29}$$

must be satisfied, and to create negative acceleration of the snakeboard, the flywheel can be stopped entirely and the effect of gravity can be utilized.

In particular, let $\xi = \xi_0\sin(\nu p)$. Then $\xi'' = -\nu^2\xi$. Therefore, if $\xi > 0$, negative angular acceleration of the flywheel that satisfies inequality (4.29) is required for upward motion along an incline, and if $\xi < 0$, positive angular acceleration is required. At the same time, if $\xi = 0$, it is not possible to satisfy inequality (4.29). In addition, this inequality cannot be satisfied in a small neighbourhood of the point where $\xi = 0$. Therefore, when the snakeboard does not have any initial velocity in the vicinity of this point, it will slide downward under the effect of gravity, despite the effect of the maximum permissible absolute value of the angular acceleration of the flywheel, until $|\xi|$ reaches a value that is large enough for there to be the possibility of satisfying inequality (4.29). After this, whenever the snakeboard begins an upward motion, it re-enters the vicinity of the point where $\xi = 0$, obtains negative acceleration under the influence of gravity, begins to descend, and so on. In other words, this example shows that in the vicinity of a point for which $\xi'' = 0$, there is a danger of self-excited oscillations of the snakeboard occurring. This nuisance can be avoided in two ways. One way is to "run" past such points by building up sufficient speed in the section where $|\xi''|$ takes a large value. The other way is to design a trajectory for ascending from curves that have a fairly large value of $|\xi''|$ with essentially instantaneous reversal of the sign of ξ'' on passing through the point where $\xi = 0$. For example, they can be arcs of circles with a fairly small radius of curvature that is consistent with the maximum permissible absolute value of the angular acceleration of the flywheel.

4.3c. Because of the possibility mentioned in Section 4.3b of the appearance of self-excited oscillations of the snakeboard on an inclined plane, it is more convenient to perform the complex manoeuvres prescribed for controlling the centre of the crossbar on a horizontal plane. As an example, we shall obtain mathematical formulae for the motion of point C when $\vartheta = 0$ along a figure-of-eight, given by the equations

$$\xi = r\cos p, \quad \eta = r\sin 2p, \quad r = \text{const}; \quad \frac{1}{\rho_\upsilon} = \frac{2\cos p(1+2\sin^2 p)}{r(\sin^2 p + 4\cos^2 2p)^{3/2}}$$

We shall assume that at the initial instant of time $p = 0$, point C has the coordinates $\xi_c = r$, $\eta_c = 0$, and the crossbar is oriented perpendicular to the $O\xi$ axis so that $\psi(0) = \pi/2 + 0$. Then, in the case of acceleration, the angular acceleration of the flywheel must be negative for $\xi > 0$ and positive for $\xi < 0$. The angles of rotation of the wheel axes are specified by the formulae

$$\operatorname{tg}\varphi_{2s} = -\operatorname{tg}\varphi_{1s} = \frac{a}{\rho_\upsilon}$$

They vanish only at the point where $\xi = \eta = 0$. We see that the formulae for designing the control of the wheel axes are not simple. The control design problem is further complicated by the fact that the current value of p can be determined only by numerical integration of differential equation (4.23).

5. Achieving maximum velocity

A simpler version of the control design can be obtained if the figure-of-eight is composed of two circles of identical radius r. Let one of the circles have its centre at the point with coordinates $\xi = r$, $\eta = 0$, and let the other circle have its centre at the point with coordinates $\xi = -r$, $\eta = 0$. These circles touch at the point where $\xi = \eta = 0$. The parametric equations of the curve as a whole have the form

$$\xi = \begin{cases} r(1+\cos p), & 0 \le p \le \pi \\ -r(1+\cos p), & \pi < p \le 3\pi, \\ r(1+\cos p), & 3\pi < p \le 4\pi \end{cases} \quad \eta = r\sin p; \quad \frac{1}{\rho_\upsilon} = \begin{cases} 1/r, & 0 \le p \le \pi \\ -1/r, & \pi < p \le 3\pi \\ 1/r, & 3\pi < p \le 4\pi \end{cases} \tag{5.1}$$

5.1. The control problem

Point C on the crossbar should move along the trajectory (5.1) on a horizontal plane ($\vartheta = 0$). At the initial instant of time $t = t_0$ point C has coordinates $\xi_0 = 2r$, $\eta = 0$. The Cx axis is oriented in the direction of the $O\eta$ axis, and the Cy axis is in the direction opposite to the $O\xi$ axis. The linear velocity of point C at the initial instant of time has the magnitude $\upsilon_c = \upsilon_0$ and is directed along the Cx axis. It is required that the velocity of point C reach its maximum value at the time $t = t_f$ of completion of its motion along the figure-of-eight, at which the robot arrives back at its initial position. At the time t_f the angular velocity of the flywheel should be equal to zero. The angular position of the flywheel obtained at the time t_f is of no significance.

5.2. Solution

We will restrict the permissible values of the angular acceleration of the flywheel to the range

$$-\varepsilon \le \ddot{\varphi} \le \varepsilon$$

Because the control is linear,[7] the maximum velocity increment is achieved for bang-bang control of the flywheel: $\ddot{\varphi} = \pm\varepsilon$. Therefore, the requirement that the angular velocity of the flywheel must be equal to zero at the end of the crossbar acceleration manoeuvre means that the sum of the time intervals when $\ddot{\varphi} = \varepsilon$ must be equal to the sum of the time intervals when $\ddot{\varphi} = -\varepsilon$.

Equations (4.20) together with (4.22) and (4.23), which describe the control process under the conditions of an increase in velocity, have the form

$$\frac{d\upsilon_c}{dt} = \frac{J_g\varepsilon}{r\mu^2}, \quad \mathrm{tg}\,\varphi_{2s} = -\mathrm{tg}\,\varphi_{1s} = \frac{a}{\rho_\upsilon}, \quad \dot{p} = \frac{\upsilon_c}{r}, \quad \mu^2 = M + \frac{b}{r^2} \tag{5.2}$$

Thus, the angles of the wheel axes are piecewise-constant and specify the radius of curvature $\rho_v = \pm r$ of the trajectory. The centre of curvature coincides with points having coordinates $\xi = r$, $\eta = 0$ and $\xi = -r$, $\eta = 0$, respectively. We will use τ to denote the time of action of the non-zero acceleration of the flywheel on the portion of the trajectory at $0 \le p \le \pi$, and we will use $\tau_1 > 0$ to denote the difference between the time of action of non-zero acceleration when $\pi < p \le 3\pi$ and the time τ. In addition, let

$$\tau^* = \frac{2\upsilon_0 r\mu^2}{J_g\varepsilon}, \quad T^2 = \frac{2\pi r^2\mu^2}{J_g\varepsilon}$$

The time τ_1 of additional work of the flywheel when $\pi < p \le 3\pi$ should be compensated when $3\pi < p \le 4\pi$. Since the effect of non-zero acceleration of the flywheel of either sign is restricted to the respective segment where the sign of the radius of curvature of the trajectory is constant, the values of τ and τ_1 are constrained by the system of inequalities

$$\tau^*\tau + \tau^2 \le T^2, \quad (\tau^* + 2\tau)(\tau + \tau_1) + (\tau + \tau_1)^2 \le 2T^2, \quad (\tau^* + 4\tau)\tau_1 + 3\tau_1^2 \le T^2$$

If we take into account that, according to the meaning of the problem, $\tau \geq 0$ and $\tau_1 \geq 0$, this system can be represented in the equivalent form

$$0 \leq \tau \leq \gamma_1, \quad 0 \leq \tau + \tau_1 \leq \gamma_2, \quad 0 \leq \tau_1 \leq \gamma_3 \tag{5.3}$$

where

$$\gamma_m = \frac{-[\tau^* + 2(m-1)\tau] + \sqrt{[\tau^* + 2(m-1)\tau]^2 + 4mT^2}}{2[m^2 - 3(m-1)]}, \quad m = 1, 2, 3$$

The quantity $\tau + 2\tau_1$ is proportional to the magnitude of the velocity acquired during complete passage along the figure of eight. It is not difficult to see that the functions $\gamma_2(\tau)$ and $\gamma_3(\tau)$ decay monotonically as τ increases. Therefore, the value $\tau = 0$ will be optimal with respect to the maximum velocity attained. In addition, the inequality $\gamma_2(0) > \gamma_3(0)$ holds. Therefore, the optimal value is

$$\tau_1 = \bar{\tau}_1 = \frac{-\tau^* + \sqrt{\tau^{*2} + 12T^2}}{6} \tag{5.4}$$

We find the maximum value of the velocity of point C at the end of the manoeuvre

$$\upsilon = \upsilon_0 + \frac{2J_g\varepsilon\bar{\tau}_1}{r\mu^2}$$

The quantity $\bar{\tau}_1$ is defined by Eq. (5.4). Summarizing the result, we present a synthesis of the optimal control law for the flywheel

$$\ddot{\varphi} = \begin{cases} 0, & 0 \leq p < \pi \\ \varepsilon, & \pi \leq p < p_1, \quad p_1 = \pi + \dfrac{\upsilon_0\bar{\tau}_1}{r} + \dfrac{J_g\varepsilon\bar{\tau}_1^2}{2r^2\mu^2} < 3\pi \\ 0, & p_1 \leq p < 3\pi \\ -\varepsilon, & 3\pi \leq p \leq 4\pi \end{cases} \tag{5.5}$$

The combination of expressions (5.5) and (5.2) comprises the optimal control law for the snakeboard as a whole on the chosen trajectory (5.1).

It may seem surprising that there is absolutely no control of the flywheel on the first portion of the trajectory at $0 \leq p < \pi$. This is especially strange when $\upsilon_0 = 0$. Then point C cannot move at all from its position. In the case when $\upsilon_0 = 0$, the control law (5.5) should be regarded as the limiting law as $\tau \to 0$. In practice this means that angular accelaration should be imparted to the flywheel when $\upsilon_0 = 0$ over a very short time interval, and there should then be a pause when point C enters the portion of the trajectory with negative curvature. The peculiarity of the optimal control just noted is attributed to the fact that as the entry velocity onto the segment with negative curvature increases, the time for control of the flywheel on this segment decreases. The total times the point C resides on the segments with positive and negative curvatures should be equal. This peculiarity of the optimal control will not be observed if the initial position of point C is placed at the origin of coordinates.

We will use k to denote the number of a single cycle of motion along the figure-of-eight. Then, the series of values of τ_k^*, corresponding to the initial cycle entry velocities, can be written as a recurrence relation

$$\tau_{k+1}^* = \tau_k^* + 2\bar{\tau}_1 = \frac{2\tau_k^* + \sqrt{\tau_k^{*2} + 12T^2}}{3}$$

The series obtained diverges as $k \to \infty$. Hence it follows that we can obtain any assigned velocity by applying the control law obtained in a series of motion cycles in the figure-of-eight. Only the danger of the wheels slipping on the supporting plane can prevent this.

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